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In the mid2030s, the health of the baby boomers will have deteriorated and many in these large cohorts will be in need of formal and/or informal long-term care.

This "**care wave**" will transform two generations: the baby boomers in need of care and their children who may supply care. It will have significant implications for labour supply, especially for women, saving behaviour, and therefore for productivity, economic growth and its inclusiveness.

The overarching objective of BB-Future is to understand the size and the implications of the care wave on economic and social outcomes, to appreciate the quality of this second ageing-related transformation and to develop policy recommendations for advance planning on the EU and Memberstate levels.

This deliverable is a scientific paper on one of two micro models that describe how care choices are determined within a family. In this first model, one child among several siblings is designated early on to take care of the parent while the other siblings do not take up any care responsibility.

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Designated child model

Ivo Bakota, Daniel Barczyk, Matthias Kredler, Fan Yang

Abstract

This working paper describes the designated-child model of the long-term care provision. The model expands on the previous work in Barczyk & Kredler (2018) by considering the children heterogeneity in the family model. We explicitly model multiple children which can potentially provide care to the parent in need. In the beginning of the work-life, children with heterogeneous characteristics form expectations over the future, and decide which child will be ”designated” as the potential care-giver. This child enters the Barczyk & Kredler (2018) game with the parent, potentially receives quid-pro-quo and altruistically motivated transfers. The non-designated children continue their life facing the labor market risks, but do not enter into bargaining with the parent regarding the long-term-care provision. Non-designated children still receive bequests once the parent dies. The introduced extensions of the Barczyk & Kredler (2018) model allow us to study the decision of how the primary informal caregiver is selected among the children. Moreover, it also allows us to analyse how the family size and characteristics influence the dynamics of the informal care provision, location choice of the children, inter-vivo transfers and the size of bequests.

Keywords: LTC
JEL codes: E00, E21, G12

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1 Introduction

One of the major challenges of the aging societies is not only the decrease of the ratio of working-age and elderly population, but also the raising share of the elderly individuals who require long-term care (LTC) (DGEFA's (2024) and Colombo et al.'s (2011)). Much of this care is provided informally [1] by relatives of the care recipients, more specifically by their children (Hoffmann & Rodrigues 2010). [2] Therefore, a better understanding of the selection of the informal care providers and the economic implications of this choice is valuable for societies facing this increasingly salient problem. This paper aims to construct a unified dynamic and stochastic economic model with multiple non-cooperative children as potential informal care providers to their parent(s).

The workhorse-model of the informal LTC provision; Barczyk & Kredler (2018) assumes a representative child, who, besides standard economic decisions, enters into bargaining with the parent over potential provision of care. While this assumptions enables the study of a wide range of questions pertaining the informal LTC and has allowed for a breakthrough in the literature, it does not allow the study of how the sibling who provides care to the parents is selected. Moreover, it does not allow the analysis of how the family size and structure affect the outcomes of interest.

It has been documented that most of the informal LTC is provided by a primary caregiver, and that the informal care provision is usually not shared equally between different family members (Barczyk & Kredler 2018). Our model will allow the analysis of how this primary caregiver is selected, which is interesting from both economic and sociological aspects. Some empirical research on which children are more likely to be selected as a primary caregiver has been conducted (Vergauwen & Mortelmans (2021), Byrne et al. (2009), Brsch-Supan (1991), Borsch-Supan et al. (1992), Borsch-Supan et al. (1996), Hiedemann & Stern (1999), Checkovich & Stern (2002), Pezzin et al. (2008), Fontaine et al. (2009)),

[1] Among analysed 19 OECD countries, about 60% of older people reported receiving only informal care (Rocard and Llena-Nozal, 2022).

[2] DGEFA (2024) also states that "The pressure for increased public provision and financing of long-term care services may grow substantially in the coming decades, especially in Member States where the bulk of long-term care is currently provided informally."

but without a structural, dynamic and forward-looking model it is difficult to escape the endogeneity of the variables considered. For example, a commonly found correlation is that the sibling that lives close to the parent is much more likely to provide LTC. [3] However, the location is obviously an endogenous variable, i.e. a decision of the children, which possibly could have been made precisely because the child anticipates providing care to the parent, and chooses to live close by. Without a structural model, correcting for the endogeneity is hard and valid instruments are quite difficult to find (Fontaine et al. 2009). Therefore, our model helps us in analysing which child is more likely to choose to live close to the parent and provide care. Our paper is also related to the macroeconomic literature which uses structural models to evaluate the effects of various policy reforms, such as Attanasio et al. (2011); Braun et al. (2019); Ko (2021), Hansen et al. (2014), Koreshkova & Lee (2021) and De Nardi et al. (2016).

Furthermore, a novel element in our model is the possibility of studying the trade-off between intergenerational support and location choice? Being close to the parent means to obtain support from the parent especially early on in the life-cycle (financial transfers, childcare, co-residence and later on of being able to potentially provide care to the parent. But, living closer limits career opportunities and perhaps comes with additional costs (e.g. needing to pay for childcare, higher housing costs).

Our model assumes that the children are not cooperative. This is in line with some evidence that the children might act more in line with the non-cooperative, rather than with the cooperative framework (Bergeot 2024). Furthermore, we also assume that the decision of who is designated as a caregiver is made rather early in life. We make these assumptions to make the model tractable, but also to make the effects of different economic mechanisms more apparent. The assumptions are also on a different end of the spectrum compared to the cooperative children model (WP2024). There are some significant differences between the cooperative and non-cooperative models. Firstly, being designated as caregiver has more severe economic consequences. The designated child will not be monetarily compensated

[3]There is also a strand of literature that finds that the living arrangements of the elderly parents and adult children are responsive to the informal provision of LTC. For example see: Mommaerts (2018) , Orsini (2010) , Engelhardt & Greenhalgh-Stanley (2010), Hollingsworth et al. (2022).

by the other children, and is more vulnerable to shocks in the labor market. This constitutes a much more risky environment compared to the case where other siblings can be used as a source of insurance for various economic shocks in life. Secondly, laws and social conventions in various countries mandate that all children, including the non-designated, need to get a slice of bequests. This can lead to the situation where a parent might need to rely more heavily on direct transfers, instead of bequests to incentivize informal LTC. Thirdly, there is a possible externality (public good) problem, where the designated child would not consider the full family costs of refusing to provide informal LTC. If a child is an only child (or a member of a cooperative siblings household), they would have an economic incentive to avoid the parent using an expensive formal LTC (i.e. nursing homes). This is because the parent would have to run down their assets to pay for the formal care, and would bequeath less to the child. However, when there are more children in the family, the bequests are split between all the children, and the designated child will not internalize all the costs for the reduced bequests to the other children, thus making this incentive weaker. This might result in a reduced quantity of informal LTC provided in equilibrium.

Our model also allows us to analyse how the institutional settings in different countries influence the outcomes relevant for the LTC. Different countries are heterogeneous when it comes to demographics, such as the average number of children. Furthermore, differences in the intensity of family ties and average geographical distance [4] of children and parents can drive the differences in outcomes in various countries. Moreover, laws governing bequests and how these have to be shared among the siblings are also not homogeneous among countries, and can be important in the interplay of care provision, gifts and bequests. All of the mentioned differences can significantly change the LTC outcomes. One of the purposes of building a quantitative, calibrated designated-child model of care provision is precisely to quantify the effects of these various institutional differences. The model can also be used to simulate various policy reforms. Besides the described institutional heterogeneity in the EU, family dynamics could be better approximated by the designated child model in some, and by the cooperative siblings model in other countries. It is therefore important to understand the outcomes and the economic logic behind the decisions in both

[4] The distance of the children and parents can also be governed by the different local labor market opportunities.

settings.

We build our model in two stages, first one being the designation stage, which happens at the start of the model life and second one being the life cycle from Barczyk & Kredler (2018). After the designation stage, the designated child enters a Barczyk & Kredler (2018) game with a parent, where they bargain about the provision of the informal care, and can receive quid-pro-quo transfers in exchange for the care provided, as well as the altruistically motivated gifts. Non-designated children continue their life without entering negotiations with the parent or the siblings, but simply faces the non-insurable risks in the labor market. Both designated and non-designated children expect the bequests from the parent.

At this point, we focused on the infinite-horizon problem. This is the limit of the final horizon problem, which is the final aim of the project. The current result is that we verify that it is indeed the less productive child that gets designated as a potential care provider.

The rest of the paper is organized as follows: sections 2 and 3 describe the problem of the non-designated and designated child, section 4 describes the designation stage, while section 5 shows the current preliminary results.

2 Model Setup - Non-designated child

At $j = 0$ (designation stage) parent enters parenthood with y^p (parent's income) and a^p (parent's assets). (At least) two children [5] are matched to parent, whose care costs θ_1 and θ_2 are drawn i.i.d. from distribution $F_\theta(\cdot|y^p)$. The family designates a caregiver $i = i^*(y^p, a^p, \theta_1, \theta_2) \in \{1, 2\}$. Children draw initial y^i from $F_{y0}(\cdot|y^p, \theta^i))$. For $j \in [0, J)$: Parent and designated child i enter Barczyk & Kredler (2018) game, while the non-designated child $-i$ has a separate problem, and receives no gifts. At the time of the parents death: $t_d \in (0, J]$, share ϵ of parent's assets a^p goes to the designated child i, while the share $(1 - \epsilon)$ goes to the non-designated child $-i$.

The important features that should be captured by our model are:

[5] For the purposes of simplified expositions, we assume there are two children in the following equations.

1. Parents should have some value from leaving bequests also to non-designated children.

2. Non-designated children have a similar (comparable) problem (risks etc.) to the designated child. This way there is a sound basis for the designation choice.

3. Non-designated children with high-y^p parents should expect larger bequests than those with low y^p.

We employ a simple strategy that accomplishes this and is described by the following procedure:

- Solve parent-i problem (Barczyk & Kredler (2018) game). In particular:
  - Take bequest value to i as an approximation for bequest value to $-i \Rightarrow$ Can solve Barczyk & Kredler (2018) game as usual with six dimensional state.
  - Treat altruism externality from $-i$ in flow utility as fixed sequence.

  $\Rightarrow$ Can solve without knowledge of $-i$'s state, even without knowledge of number of other children.

  $\Rightarrow$ Approximate bequest distribution from this problem by $b^{-i} = (1-\epsilon)a^p_{t_d} \simeq \ln \mathcal{N}(m_b, \sigma_b)$, where σ_b is a fixed number and where m_b varies according to family's initial conditions: $m_b = m(y^p, a^p_0, \theta^i)$.

- Solve $-i$'s problem, with states (j, y^{-i}, a^{-i}, m_b), feeding in log-normal bequest distribution with different means.

- At $j = 0$: For any combination $(y^p, a^p_0, \theta_1, \theta_2)$, obtain starting values V^p_0, V^i_0, V^{-i}_0 for both assignments $i \in \{1, 2\}$. Chose the "better" one according to the assignment criteria to get assignment policy $i^*(y^p, \theta_1, \theta_2)$.

2.1 Non-designated child's problem

The non-designated child maximizes their lifetime utility:

$$V^{-i}(0,a,y) = \max_c E\left[\int_0^J e^{-\rho j} u\left(c(j)\right) dj + e^{-\rho J} V(0,a,y)\right] \tag{1}$$

.

Where c is consumption, J is the age of death, ρ is the discount factor, u is a utility function with standard properties, a are assets, y is labor productivity, $-i$ is the non-designated child and $V^{-i}(j,a,y)$ is a value function given age, assets and labor productivity.

The problem for a child with a dead parent is the same, no matter if the child gave care in the past or not. In that case, the value function $V^{aln}(j,a,y)$ solves

$$\begin{aligned} -V_j^{aln} + \rho V^{aln} = \max_c \left\{u(c) + (ra + g_j y - c)V_a^{aln}\right\} + \sigma^2 a^2 V_{aa}^{aln} \\ + \underbrace{\sum_{y' \neq y} \eta_{yy'} \left[V^{aln}(j,a,y') - V^{aln}(j,a,y)\right]}_{\text{jump terms in } y}, \end{aligned} \tag{2}$$

where the superscript aln denotes the non-designated child whose parent has dies, and g_j is a Mincer profile for earnings (labor productivity).
$\Rightarrow$ Solved backward on $j \in [0, J)$ given terminal value $V(0,a,y) = V^*(a,y)$, where $V^*(a,y)$ is the value of entering the parent stage before being matched to children.

Non-designated child value $V^{-i}(j, a^{-i}, y^{-i}, m_b(, y^p))$ solves

$$\begin{aligned} -V_j^{-i} + \rho V^{-i} = \max_c \left\{u(c) + (ra^{-i} + g_j y^{-i} - c)V_a^{-i}\right\} + \sigma^2 (a^{-i})^2 V_{aa}^{-i} \\ + \sum_{y' \neq y} \eta_{yy'} \left[V^{aln}(j,a,y') - V^{aln}(j,a,y)\right] \\ + \delta(j, y^p) \int_0^\infty \left[V^{k,aln}(j, a^{-i} + b, y) - V^{-i}(j, a^{-i}, y)\right] dF_b(b|m_b), \end{aligned} \tag{3}$$

where child takes as given

- $\delta(j, y^p)$: parent's death hazard

- $F_b(b|m_b)$: cdf of bequest b given m_b
- Can simplify and assume that hazard δ is only function of j for now, otherwise have another state y^p.

3 Model Setup - Designated Child

This section illustrates the economic problem between the parent and the designated child. The setup closely follows (Barczyk & Kredler 2014) and (Barczyk & Kredler 2018). For now, we switch off the parent's death risk and focus on the infinite horizon to study the factors affecting care arrangement. But in the final version, we will incorporate death hazard and adopt a finite horizon model, where age (j) will also be a state variable.

3.1 State Space

The family consists of a parent and a child. The parent cares about the child but not vice versa, resulting in single-direction altruism. Both agents are infinitely lived.

The family state is given by the vector $z = (a^k, a^p, y^k, y^p, s^p)$, where the superscript $i \in \{k, p\}$ corresponds to kid and parent, respectively. $a^i \geq 0$ denotes the wealth of player i. y^i is the labor income following a Poisson process. $s^p \in \{0, 1\}$ indicates the parent's disability status, signaling the need for care. Specifically, $s^p = 0$ stands for healthy while $s^p = 1$ indicates that the parent is disabled.

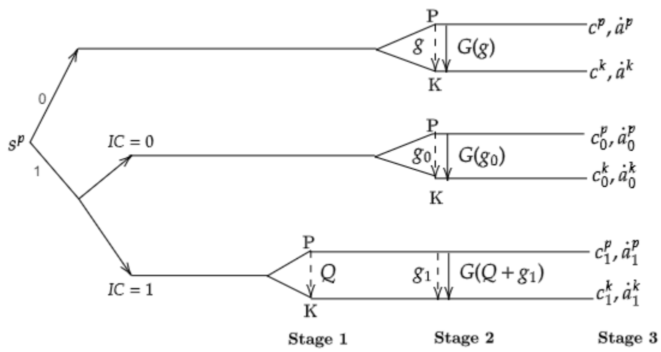
3.2 Timing

The following three stages happen sequentially over a Δt period:

Stage 1. **Disabled** parent ($s^p = 1$) and the designated kid determine which care option to take, and the level of exchange-motive transfer Q for informal care. If informal care is chosen, the transfer Q is determined through Nash bargaining.

Stage 2. Parents decide the altruistic gift flow $g \geq 0$. Then, a convex transaction cost $(G())$ applies to the total parent transfer $Q + g$

Stage 3. Given the care arrangement and total transfer, parents and kid simultaneously decide their optimal consumption and saving.



3.3 Preference

We model the per-period felicity of each player using a CRRA utility function:

$$u^i(c^i) = \frac{(c^i)^{1-\gamma}}{1-\gamma}; i \in \{k, p\} \tag{4}$$

where $\gamma > 0$ governs the relative risk aversion and c^i denotes player i's consumption. The kid's flow utility is given by equation (4), which depends only on his own consumption. However, we assume the parent's flow utility is given by $u^p(c^p) + \alpha^p u^k(c^k)$, which depends on both her own utility and her kid's. α^p determines the degree of altruism. Both agents discount the future utility at rate $\rho > 0$.

3.4 Flow Budget Constraints

This section presents the consolidated budget constraint over the time period Δt for two agents.

$$da^k = (ra^k + y^k + g^p - c^k - \delta^k y^k \mathbb{h} + Q\mathbb{h})dt + \sigma a^k dB \tag{5}$$

In each time period, the kid gets interest income from his wealth, an altruistic gift from the parent, and labor income. Moreover, if the kid provides informal care ($\mathbb{h} = 1$), he will lose part of his labor income ($\delta^k y^k$) but get an exchange-motive transfer (Q) from the parent. From these resources, the kid determines the optimal consumption and savings. Additionally, the flow budget includes a geometric Brownian motion of the wealth.

$$da^p = (\underbrace{ra^p + y^p - p^m s^p(1-\mathbb{h})}_{x^p} - G(g^p + Qs^p\mathbb{h}, x^p) - c^p)dt + \sigma a^p dB \tag{6}$$

When the parent is healthy ($s^p = 0$), she earns the labor income and collects interest income from her savings. From these resources, she decides how much to consume and how much to transfer altruistically to the child. If the parent is disabled and chooses formal care ($s^p = 1, \mathbb{h} = 0$), she pays for the formal care cost p^m and optimizes her decisions out of the remaining cash on hand. If the disabled parent chooses informal care($s^p = 1, \mathbb{h} = 1$), she pays the kid for caregiving and saves the formal care cost. A convex transaction cost $G(., x^p)$, which is proportional to the parent's resource, applies to the total transfer($g^p + Qs^p\mathbb{h}$),

3.5 Agents' Problem

$$-V_t^i(z) + \rho V^i(z) = J^i + H^{i,1}(z, \underbrace{V_{a^p}^p, V_{a^k}^p, V_{a^k}^k}_{V_a}), \quad i \in \{k, p\} \tag{7}$$

Equation (7) characterizes the agent's problem, where subscript of V^i denotes partial derivative, J^i consists of the jump terms (stochastic productivity, parent's disability status), and terms related to Brownian shocks.

We now derive the Hamiltonian functions $H^{i,1}(.)_{i\in\{k,p\}}$ by backward induction on the stages of the instantaneous game. We do this for the general case when the parent is disabled; if the parent is not disabled or chooses formal care, only the last two stages are relevant. We denote $H^{i,n}(.)$ as the Hamiltonian of player i in the n^{th} stage, and $x^{i,n}$ represent player i's stage-n cash-on-hand, as determined by earlier stages. The vector $x^n = [x^{k,n}, x^{p,n}]$ contains the cash-on-hand for both players.

Stage 3. Consumption-saving decision: Given the IC decision $\mathbb{h}$ and stage-3 cash on hand

x^3, both agents optimize their consumption and savings according to the Hamiltonian:

$$\begin{aligned} H^{k,3}(z,V_a|x^3,h) &= \max_{c^k}\{u(c^k)+[\dot{a}^k]^+V^{k,f}_{a^k}-[\dot{a}^k]^-V^{k,b}_{a^k}\} \\ H^{p,3}(z,V_a|x^3,h) &= \max_{c^p}\{u(c^p)+[\dot{a}^p]^+V^{p,f}_{a^p}-[\dot{a}^p]^-V^{p,b}_{a^p}\} \\ where \quad \dot{a}^i &= x^{i,3}-c^i,\ i\in\{k,p\} \\ [a]^+ &= max\{0,a\},\ [a]^- = max\{0,-a\}, \end{aligned}$$

Each agent equates the marginal cost ($u'(c^i)$) and the marginal benefit ($V^i_{a^i}$) of saving to determine the optimal consumption. Note that the marginal benefit depends on the drift direction, and so does the optimal consumption. Consequently, the agent decides the optimal saving direction by comparing three possible Hamiltonians: $H^{i,3,*}=\max\{H^{i,3}_{\dot{a}<0},H^{i,3}_{\dot{a}=0},H^{i,3}_{\dot{a}>0}\}$

Stage 2. Gift-giving stage: Due to single-direction altruism, only the parent optimizes in this stage. Given the IC decision $\mathbb{h}$ and stage-2 cash on hand x^2, the parent decides the optimal gift by solving the Stage-2 Hamiltonian, taking into account both players' best responses from stage 3:

$$\begin{aligned} H^{p,2}(z,V_a|x^2,h) &= \max_g\ u\left(c^{p,*}(z|g)\right)+\alpha^p u\left(c^{k,*}(z|g)\right) \\ &+ [x^{k,2}+g-c^{k,*}(z|g)]^+V^{p,f}_{a^k}-[x^{k,2}+g-c^{k,*}(z|g)]^-V^{p,b}_{a^k} \\ &+ [x^{p,2}-G(g+Qs^p\mathbb{h})-c^{p,*}(z|g)]^+V^{p,f}_{a^p}-[x^{p,2}-G(g+Qs^p\mathbb{h})-c^{p,*}(z|g)]^-V^{p,b}_{a^p} \end{aligned}$$

where $c^{i,*}(z|g)$ denotes the best response of agent i solved from the stage 3. Note that the parent can influence the kid's optimal choice by affecting the kid's drift through the gift. Therefore, we have nine possible optimal gifts for each state point, resulting from the different drift combinations of the two agents $(\dot{a}^k,\dot{a}^p)$. The computation of optimal gift is illustrated in Appendix B, which provides a general description by considering both types of transfers under the IC.[6]

In this stage, the kid merely receives the gift from his parent, so the kid's Hamiltonian

[6]For computing optimal gift only, one can set $Q=0$ in Appendix B.

in the second stage is simply:

$$H^{k,2}(z, V_a|x^2, h) = H^{k,3}(z, V_a|[x^{k,2} + g, x^{p,3}], h)$$

Stage 1. Bargaining stage: This stage only happens when the parent is disabled, and it determines the IC decision and the accompanying exchange-motive transfer $Q \geq 0$. The Stage-1 Hamiltonian for agent i:

$$H^{i,1}(z, V_a) = H^{i,2}(z, V_a|[(1-\mathbb{h})x^k_{fc} + \mathbb{h}(x^k_{ic} + Q^*), (1-\mathbb{h})x^p_{fc} + \mathbb{h}(x^p_{ic} - G(\tilde{g}(Q^*)))]) \quad (8)$$

where

$$\begin{aligned}
\tilde{g}(Q) &= g(Q) + Q \\
x^k_{fc} &= x^k = ra^k + y^k;\ x^k_{ic} = x^k - \delta^k y^k \\
x^p_{ic} &= x^p = ra^p + y^p;\ x^p_{fc} = x^p - p^m \\
\mathbb{h} &= \begin{cases} 1 & \text{if } \exists Q \geq 0 \text{ s.t. } S^p(Q) \geq 0 \ \&\ S^k(Q) \geq 0 \\ 0 & \text{otherwise} \end{cases} \\
\text{where } S^i(Q) &= H^{i,2}(z, V_a|[y^k_{ic} + Q, y^p_{ic} - G(\tilde{g}(Q))], \mathbb{h} = 1) - H^{i,2}(z, V_a|[y^k_{fc}, y^p_{fc}], \mathbb{h} = 0) \\
Q^* &= \begin{cases} arg\max_{Q \geq 0}\{S^k(Q)^{\omega} S^p(Q)^{1-\omega}\} & \text{if } \mathbb{h} = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Informal care occurs if there exists a non-negative transfer $Q \geq 0$ that generates positive surplus for both agents. We define the surplus as the difference in the Hamiltonians between IC and formal care. To assess whether choosing IC can improve efficiency, we calculate both players' reservation Q, at which their surplus is zero. IC is considered attractive if $\bar{Q}^p$, the maximum Q that the parent is willing to pay for IC ($S^p(\bar{Q}^p) = 0, \partial_Q S^p(Q) < 0$), exceeds $\underline{Q}^k$, the minimum Q that kid willing to take and provide IC ($S^k(\underline{Q}^k) = 0, \partial_Q S^k(Q) > 0$). Appendix A provides more details on computing the cutoffs. If IC is chosen, the optimal Q^* is determined by the Nash bargaining between two agents.

3.6 Thought Experiment: IC Determination

As previously described, informal care (IC) is selected if a non-negative transfer exists that yields positive surpluses for both players. This section presents a simple thought experiment to study factors relating to the likelihood of IC. Suppose the drift status under the two care options is the same, we can rewrite the surplus for the two agents as follows:

$$
\begin{aligned}
S^p &= (\dot{a}_1^p - \dot{a}_0^p)V_{a^p}^p + (\dot{a}_1^k - \dot{a}_0^k)V_{a^k}^p \\
&= \left[-G(\tilde{g}_1) - (-p^m - G(g_0))\right] V_{a^p}^p + (\tilde{g}_1 - \delta^k y^k - g_0)V_{a^k}^p \\
S^k &= (\tilde{g}_1 - \delta^k y^k - g_0)V_{a^k}^k
\end{aligned}
$$

where subscript of transfer stands for the IC status (0 for FC, 1 for IC). In summary, we can express the competing forces determining IC in terms of monetary gain or loss:

IC=0				
	parent	$-(p^m + G(g_0))$	kid	g_0
IC=1				
	parent	$-(G(\tilde{g}_1))$	kid	$\tilde{g}_1 - \delta_k y^k$

IC would happen iff $\begin{cases} G(\tilde{g}_1) \leq p^m + G(g_0) \\ \tilde{g}_1 - \delta_k y^k \geq g_0 \end{cases}$ That is, switching to informal care is beneficial if i) the parent's care expenditure gets reduced and ii) the kid has more net monetary gains. Therefore, the likelihood of IC is:

1. positively correlates with formal care cost: $\uparrow p^m \rightarrow \uparrow Pr(IC)$
2. negatively correlates with kid's productivity loss or care cost: $\uparrow \delta^k or \uparrow y^k \rightarrow \downarrow Pr(IC)$

3.7 To-do List

1. Add heterogeneous kid's care cost
2. *Optimal Q^*:* Currently use the weighted average of efficiency layer as in (Barczyk & Kredler 2018) (e.g., $Q^* = \omega \bar{Q}^p + (1-\omega)\underline{Q}^k$), which has not adjusted for single-altruism and asymmetric transfer expenditure.

4 Determination of Caregiver

Given the value functions solved in the previous two sections, we now determine the caregiver ($n^* \in \{1, 2\}$) by maximizing the family value function:

$$
\begin{aligned}
n^* = arg \quad \max_n \quad & V^{fam,n}(a^p, y^p, y^{k,n}, y^{k,-n}) = ... \\
... \qquad & V^p(a^{k,n} = 0, a^p, y^{k,n}, y^p) + V^k(a^{k,n} = 0, a^p, y^{k,n}, y^p) + V^{nd}(a^{k,-n} = 0, y^{k,-n})
\end{aligned}
$$

where $V^{fam,n}$ denotes the family's value of designating kid n. Note here that the value V^k in the last line captures the value of the designated child n that is paired with the parent and $V^{nd}(= V^{-n})$ captures the value of the non-designated child that parts ways with the family. Since the designation stage occurs at the initial phase, we assume both kids start with zero wealth. Thus, a family is characterized by the states of two kids' entry productivity($y^{k,n}, y^{k,-n}$), parent's wealth (a^p) and productivity(y^p).[7]

The family value function is defined as the sum of three players' value functions. In particular, due to the two distinct potential caregivers, each family has two values ($V^{fam,1}$, $V^{fam,2}$), from which we pick the maximum to designate the caregiver.

5 Numerical Results

For now, we compute and analyze the results based on an infinite-horizon setup, but there will eventually be a finite horizon incorporating the parents' death risk and their bequest decision.

5.1 Parameter Value

According to the parameter values, the kid will lose $\delta^k y^k = 0.3 * [25, 50] = [7.5, 15]$ if he provides informal care. To some extent, this productivity loss represents the minimum compensation the parent must pay the child for informal care. Since the formal care cost falls in the middle of this productivity loss range, informal care is more likely to be chosen when the child's productivity (and thus the required compensation) is lower.

[7]For now, we set the bequest expectation of non-designated child to zero due to infinite-horizon setup.

Table 1: Parameters in Numerical Example

Parameter	Description	Value	Parameter	Description	Value
α^p	altruistic level	0.25	γ	coeff.risk aversion	2
p^m	formal care cost	10	δ^k	kid's prod. loss rate	0.3
$y_h^i, y_l^i, i \in \{k, p\}$	labor income	25,50	ξ	Poisson rate	0.1
ρ	discount rate	0.05	σ	risk in saving	0.02
r	interest rate	0.03	τ	transaction cost	1
ω	bargaining weight	$1e^{-5}$			

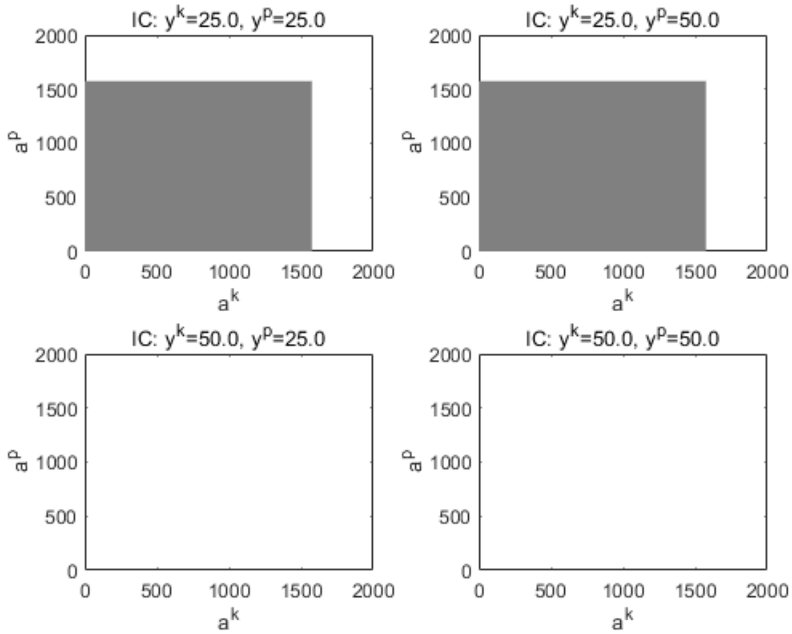


Figure 1: IC Arrangement

5.2 Informal Care Arrangement

The shadow area in Figure 1 indicates where informal care is chosen. The informal care only happens when kid is less productive($y^k = 25$). When the kid's productivity is high, the parent chooses formal care. This pattern aligns with the reasoning of minimum compensation mentioned earlier: a more productive kid needs a proportionally higher payment, which diminishes the worthiness of IC.

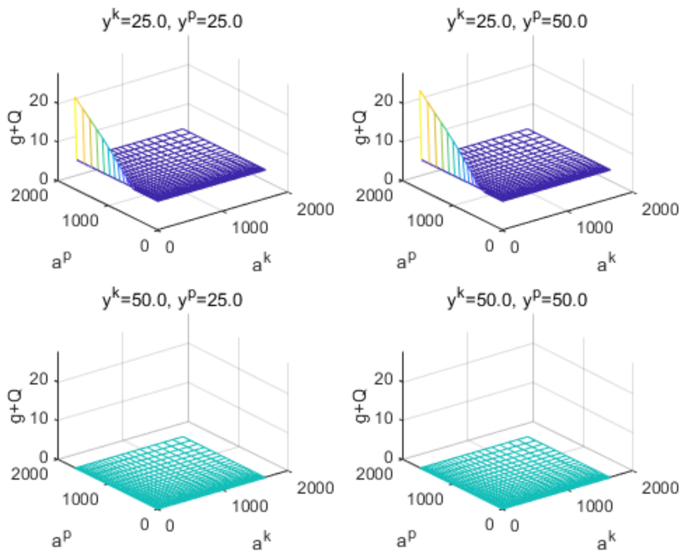


Figure 2: Aggregate Transfer $\tilde{g} = g + Q$

	Benchmark	$\uparrow \delta^k$ 15%	$\downarrow p^m$ 15%	$2{\times}\alpha^p$
$Pr(IC)$	0.5	0.394	0.348	0.5
$\bar{g}$	3.757	3.399	2.62	3.93

Table 2: Changes in IC-share and Average Transfer

5.3 Aggregate Transfer

Figure 2 plots the aggregate transfer- the sum of exchange motive Q and altruistic gift g- when the parent is disabled. As previously noted, IC does not happen when the kid is productive(Figure 1), which implies $Q = 0$. Furthermore, the altruistic gift is also zero when the kid's productivity is high, resulting from the lower transfer motive.

When the kid's productivity is low, the IC occurs, and most of the aggregate transfer is at the kid's reservation level ($\underline{Q}^k$) due to the kid's limited bargaining power. When the kid is liquid-constrained and the parent is wealthy, the transfer motive out of altruism complements, leading to a higher aggregate transfer.

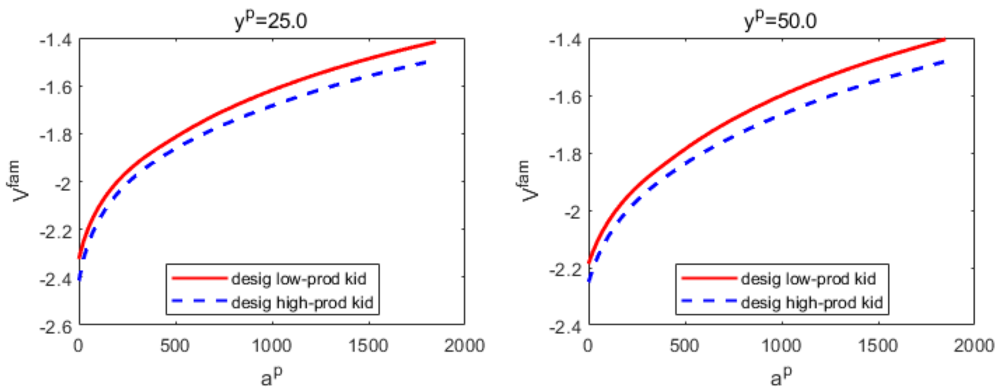


Figure 3: Family Value Function

5.4 Comparative Static

Table 2 presents comparative statics examining the changes in IC share and average transfer in response to three factors. When the kid's opportunity cost is 15% higher, the IC share and the aggregate transfer decrease. A similar pattern is observed when we decrease the formal care cost. Doubling the altruism level does not affect IC share but raises the parent's transfer motive, leading to a higher average transfer. Overall, these numerical results align with the qualitative implications of the model.

5.5 Who Becomes the Designated Caregiver

Each diagram of Figure 3 plots two possible family value functions against the parent's wealth. The red curve corresponds to the family's value of designating the low-productivity kid, while the blue dashed line shows the value when the high-productivity kid gets designated. Across all levels of the parent's wealth and productivity, it is optimal to designate the low-productivity child as the caregiver, as indicated by the higher family values.

Fixing the parent's wealth level, we further examine the pattern of designating the low-productivity child by increasing the number of productivity grid points to five. For simplicity, we illustrate the following results using the productivity grid indices to represent the corresponding productivity levels. This substitution is valid since the indices and productivity levels are both monotonically increasing.

Table 3 presents all possible distinct pairs of two kids' productivity. The rows corre-

Table 3: Possible Kids' Productivity Pairs

$ind(y^{k,1})$ \ $ind(y^{k,2})$	1	2	3	4	5
1		(1*,2)	(1*,3)	(1*,4)	(1*,5)
2	(2,1*)		(2*,3)	(2*,4)	(2*,5)
3	(3,1*)	(3,2*)		(3*,4)	(3*,5)
4	(4,1*)	(4,2*)	(4,3*)		(4*,5)
5	(5,1*)	(5,2*)	(5,3*)	(5,4*)	

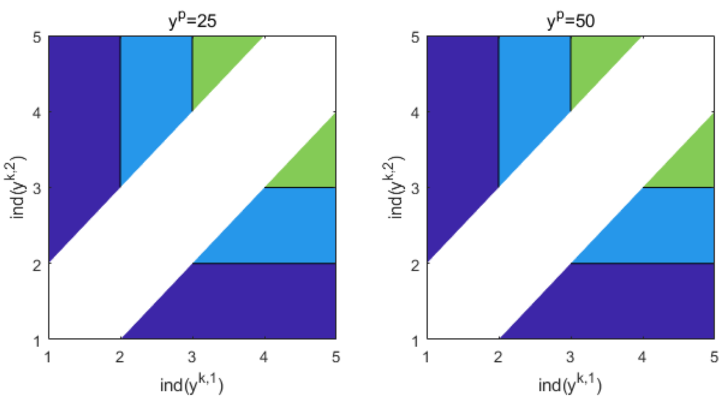


Figure 4: Designated Child

spond to child 1's productivity, and the columns are for the other child's. In each pair, the designated child is marked with an asterisk at the corresponding productivity index. For example, if kid-1 has the lowest productivity ($ind(y^{k,1} = 1)$ but kid-2's productivity is the highest ($ind(y^{k,2} = 5)$, which corresponds to the cell located at the right-top corner, kid-1 is designated as the caregiver.

Figure 4 plots the results from Table 3. Apart from the blank diagonal, where both kids have the same productivity, the designation rule is similar to a Leontief function (e.g., $ind(y^{k,n^*}) = min\{ind(y^{k,n}), ind(y^{k,-n})\}$). Thus, the low-productivity child is consistently chosen as the caregiver, confirming our findings' robustness.

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A First Stage: Surplus Function and Cutoff Qs

In this section, we derive and compute the cutoff values of exchange-motive transfer Q that make two agents' surplus zero. In particular, due to the nature of the transfer that goes from parent to child, the parent's cutoff $\bar{Q}^p$ represents the upper bound that the parent is willing to give in exchange for informal care(IC). Conversely, the kid's cutoff $\underline{Q}^k$ represents the lower bound, which makes the kid willing to provide IC. If $\bar{Q}^p \geq \underline{Q}^k$, then IC will be chosen.

Denote $H_0^i, i \in \{k, p\}$ as agent i's Hamiltonian when formal care is chosen, which serves the outside option relative to informal care. The surplus $S^i(Q)$ of player i is the difference between the Hamiltonians under the IC scenario and the formal-care scenario (e.g., $S^i(Q) = H_1^i(Q) - H_0^i$)), where the subscript denotes the IC status.

A.1 Kid: $\underline{Q}^k$

Denote kid's cash on hand under IC as $x_1^k = ra^k + (1-\delta^k)y^k$, the kid's surplus becomes:

$$S^k(Q) = \begin{cases} u(c_1^k) + \left(x_1^k + Q - c_1^k\right) V_{a^k}^{k,da_1^k} - H_0^k & if\ da_1^k \neq 0 \\ u(x_1^k + Q) - H_0^k & if\ da_1^k = 0 \end{cases} \tag{IA.1}$$

Equation (IA.1) indicates that the kid's surplus increases linearly in Q with rate $V_{a^k}^{k,da_1^k}$ if the drift is non-zero, and increases with a decreasing rate if the kid is hand-to-mouth. Thus, the root for $S^k(Q) = 0$ entails the minimum transfer level that makes the kid willing to provide IC, which we denote as $\underline{Q}^k$:

$$\underline{Q}^k = \begin{cases} -\frac{u(c_1^k) + \left(x_1^k - c_1^k\right) V_{a^k}^{k,da_1^k} - H_0^k}{V_{a^k}^{k,da_1^k}} & if\ da_1^k \neq 0 \\ \left((1-\gamma) H_0^k\right)^{1/(1-\gamma)} - x_1^k & if\ da_1^k = 0 \end{cases}$$

A.2 Parent: $\bar{Q}^p$

The parent's surplus is:

$$S^p(Q) = u(c_1^p) + \alpha_p u(c_1^k) - H_0^p + (x_1^p - G(Q) - c_1^p) V_{a^p}^{p,da_1^p} + \left(x_1^k + Q - c_1^k\right) V_{a^k}^{p,da_1^k}$$

As we can see, the cutoff $\bar{Q}$ depends on both agents' drift status (da_1^k, da_1^p), which can be categorized in four possible cases:

1. $da^p * da^k \neq 0$ Rewrite parent's surplus as:

$$S^p(Q) = \underbrace{u(c_1^{dp,p}) + \alpha_p u(c_1^{dk,k}) + \left(x_1^p - c_1^{dp,p}\right) V_{a^p}^{p,da_1^p} + \left(x_1^k - c_1^{dk,k}\right) V_{a^k}^{p,da_1^k} - H_0^p}_{\zeta_1} - G(Q) V_{a^p}^{p,da_1^p} + Q V_{a^k}^{p,da_1^k}$$

   Equating $S^p(Q) = 0$, we will get a quadratic equation of Q:

$$\underbrace{-\frac{\theta}{2} V_{a^p}^{p,da_1^p}}_{<0} Q^2 - (V_{a^p}^{p,da_1^p} - V_{a^k}^{p,da_1^k}) Q + \zeta_1 = 0 \tag{IA.2}$$

   and the larger root (e.g., $\frac{-b}{2a} + \left|\frac{\sqrt{b^2-4ac}}{2a}\right|$) will be chosen because of negative slope of S^p.

2. $da^p = 0, da^k \neq 0$: Rewrite parent's surplus as:

$$S^p(Q) = u(x_1^p - G(Q)) + Q V_{a^k}^{p,da_1^k} + \underbrace{\alpha_p u(c_1^{dk,k}) + \left(x_1^k - c_1^{dk,k}\right) V_{a^k}^{p,da_1^k} - H_0^p}_{\zeta_2} \tag{IA.3}$$

   Unlike the first case, the parent's surplus is now a general non-linear function of Q, and we will solve for the cutoff using a root-finding algorithm. Before that, we need to check the monotonicity of the surplus, which ensures the validity of the root:

$$\partial_Q S^p(Q) = \underbrace{u'(x_1^p - G(Q))(1+\theta Q) * (-1)}_{<0} + \underbrace{V_{a^k}^{p,da_1^k}}_{>0} \tag{IA.4}$$

$$\partial^2_{QQ} S^p(Q) = -1 * \left(\underbrace{(1+\theta Q)^2 u''(c_1^p(Q)) * (-1) + u'(c_1^p(Q))\theta}_{>0} \right) \tag{IA.5}$$

The first order derivative (IA.4) suggests that the monotonicity is ambiguous. On the one hand, a higher Q reduces the parent's own consumption, which lowers the surplus; conversely, due to altruism, the parent's surplus can be improved by transferring some resources to their kid. How does this partial derivative change with Q? Negative second order derivative (IA.5) confirms the parent's surplus is concave in Q.

Therefore, the surplus can either be an upside-down parabola or a decreasing concave function. We will check the partial derivative and surplus value at the boundary $[0, Q_{max} = G^{-1}(x_1^p)]$ for refining root-finding procedure.

- $\partial_Q S^p(0) < 0$: check surplus value at two bounds for corner solution; then implement Newtons Method for interior solution
- $\partial_Q S^p(0) > 0$ & $\partial_Q S^p(Q_{max}) < 0$: $\exists \hat{Q}\ s.t.\ \partial_Q S^p(\hat{Q}) = 0$, repeat above procedure between interval $[\hat{Q}, Q_{max}]$
- $\partial_Q S^p(0) > 0$ & $\partial_Q S^p(Q_{max}) \geq 0$: set $\bar{Q} = Q_{max}$?

One can also firstly solve the breakeven point of first order derivative $\hat{Q}$ (e.g.,$\partial_Q S^p(\hat{Q}) = 0$), then decide which of the following strategy to take depending on the location of $\hat{Q}$

- $\hat{Q} \leq 0$: standard root-finding for monotonic function[8]
- $\hat{Q} \geq Q_{max}$: eventually confined to $\bar{Q}$ to Q_{max}?

[8]Standard root-finding: check the function value at two bounds to rule out some corner solutions, then apply Newton Method for interior solution

- $\hat{Q} \in (0, Q_{max})$: standard root-finding for monotonic function on a refined interval $[\hat{Q}, , Q_{max}]$

3. $da^p \neq 0, da^k = 0$: Rewrite parent's surplus as:

$$
\begin{aligned}
S^p(Q) &= u(c_1^{dp,p}) + \left(x_1^p - c_1^{dp,p}\right) V_{a^p}^{p,da_1^p} - H_0^p + \alpha_p u(x^k + Q) - G(Q) V_{a^p}^{p,da_1^p} \\
\partial_Q S^p(Q) &= \underbrace{\alpha_p u'(c^k)}_{>0} - \underbrace{(1+\theta Q) V_{a^p}^{p,da_1^p}}_{>0} \\
\partial^2_{QQ} S^p(Q) &= \alpha_p u^{''}(x^k + Q) - \theta V_{a^p}^{p,da_1^p} < 0
\end{aligned}
$$

4. $da^p = 0, da^k = 0$: Rewrite parent's surplus as:

$$
\begin{aligned}
S^p(Q) &= u(x^p - G(Q)) + \alpha_p u(x^k + Q) - H_0^p \\
\partial_Q S^p(Q) &= u'(c^p(Q))(1+\theta Q) * (-1) + \alpha_p u'(c^k) \\
\partial^2_{QQ} S^p(Q) &= -1 * \left(\underbrace{(1+\theta Q)^2 u^{''}(c_1^p(Q)) * (-1) + u'(c_1^p(Q))\theta}_{>0} \right) + \alpha_p u^{''}(x^k + Q) < 0
\end{aligned}
$$

Therefore, the parent's surplus function is concave (e.g., negative second order derivative) in all possible scenarios. We will solve $\bar{Q}^p$ using the derivative-breakeven($\hat{Q}$) approach:

1. Solve $\hat{Q}$ within $[0, Q_{max}]$
2. Solve $\bar{Q}^p$ within $[\hat{Q}, Q_{max}]$
3. Confine $\bar{Q}^p$ to the well-defined upwind interval $[L, U]$

B Aggregate Transfer under Informal Care

As described in section 3.5, both exchange-motive (Q) and altruistic gift (g)are financial transfers from the parent to the designated kid. Moreover, the sum of these two transfers is subject to a convex transaction cost. In this section, we prove the relationship between these two types of transfers.

B.1 Numerical strategy for First Stage

In the gift-giving stage, given Q, parent solves the Hamiltonian for optimal gift:

$$\begin{aligned} J^p(g|Q) &= \max_g \; u\left(c_1^{p,*}(g|Q)\right) + \alpha^p u\left(c_1^{k,*}(g|Q)\right) \\ &+ \max\left(0, x_1^k + g + Q - c_1^{k,*}(g|Q)\right) V_{a^k}^{p,f} + \max\left(0, -(x_1^k + g + Q - c_1^{k,*}(g|Q))\right) V_{a^k}^{p,b} \\ &+ \max\left(0, x_1^p - G(g+Q) - c_1^{p,*}(g|Q)\right) V_{a^p}^{p,f} + \max\left(0, -(x_1^p - G(g+Q) - c_1^{p,*}(g|Q))\right) V_{a^p}^{p,b} \end{aligned}$$

where $c^{i,*}$ denotes the best response of agent i. Depending on the drift status of both players, we can solve the first order condition for the optimal gift as a function of Q :

1. $\dot{a}^p \dot{a}^k \neq 0$

$$\begin{aligned} g^*_{da_p,da_k}(Q) &= \frac{V_{a^k}^{p,da^k} - V_{a^p}^{p,da^p}}{V_{a^p}^{p,da^p}} \times \frac{1}{\theta(x_1^p)} - Q \\ g'(Q) &= \begin{cases} -1 & Q \leq \zeta \\ 0 & Q > \zeta \end{cases} \end{aligned}$$

2. $\dot{a}^p \neq 0;\ \dot{a}^k = 0$

$$\begin{aligned} g^*_{da_p,0}(Q) &= argfz \min_g F(g|Q) = \alpha^p u_c(x_1^k + g + Q) - (1 + \theta\,(g+Q))\, V_{a^p}^{p,da^p} \\ g'(Q) &= \begin{cases} -\frac{\partial F/\partial Q}{\partial F/\partial g} = -\frac{\alpha^p u_{cc} - \theta V_{a^p}^{p,da^p}}{\alpha^p u_{cc} - \theta V_{a^p}^{p,da^p}} \textbf{=-1} & \exists g \geq 0 \\ 0 & \nexists g \end{cases} \end{aligned}$$

3. $\dot{a}^p = 0;\ \dot{a}^k \neq 0$

$$\begin{aligned} g^*_{0,da_k}(Q) &= argfz \min_g F(g|Q) = V_{a^k}^{p,da^k} - u'(x_1^p - G(g+Q))(1 + \theta(g+Q)) \\ g'(Q) &= \begin{cases} -\frac{\partial F/\partial Q}{\partial F/\partial g} = -\frac{\left(u_{cc}(1+\theta\tilde{g}^2) - u_c\theta\right)}{\left(u_{cc}(1+\theta\tilde{g}^2) - u_c\theta\right)} \textbf{=-1} & \exists g \geq 0 \\ 0 & \nexists g \end{cases} \end{aligned}$$

4. $\dot{a}^p = 0;\ \dot{a}^k = 0$

$$\begin{aligned} g^*_{0,0}(Q) &= argfz \min_g F(g|Q) = \alpha^p u_c(x_1^k + g + Q) - u'(x_1^p - G(g+Q))(1+\theta(g+Q)) \\ g'(Q) &= \begin{cases} -\frac{\partial F/\partial Q}{\partial F/\partial g} = \textbf{=-1} & \exists g \geq 0 \\ 0 & \nexists g \end{cases} \end{aligned}$$

Therefore, conditional on the existence of g, one unit increase in Q reduces g^* by 1 unit. Such perfect substitutability indicates there exists a desired transfer $\tilde{g}^*$, such that if either g or Q is below this desired level, the other can top up the sum to reach $\tilde{g}^*$. According to the timing protocol of the game, the altruistic gift g serves as the top-up device for the first-stage transfer Q. In addition, $g = 0$ when $Q \geq \tilde{g}^*$.

B.2 Numerical strategy for First Stage

From **the second stage**, we have got the desired gift ($\tilde{g}^*$) and the upwind layers (LB, UB) that apply to the total transfer($\tilde{g} = Q + g$) for all nine drift statuses. In addition, we also calculate the unconstrained efficiency layer ($\underline{Q}^k, \bar{Q}^p$) for each upwind region.

Here is how we use these results for the first stage. In each of the nine regions:

1. Confine desired $\tilde{g}^*$ to the corresponding upwind layer, and compare the confined $\tilde{g}^*$ with the lower and upper bound. We can get three possible cases

   * $\tilde{g}^* \leq LB$

   ** $\tilde{g}^* \geq UB$

   *** $\tilde{g}^* \in (LB, UB)$

2. Confine unconstrained Q-cutoff according to the above three cases,

   * $\tilde{g}^* \leq LB \rightarrow sandw(LB, \bar{Q}/\underline{Q}, UB)$

   ** $\tilde{g}^* \geq UB \rightarrow Q^* = 0$

   *** $\tilde{g}^* \in (LB, UB) \rightarrow sandw(\tilde{g}^*, \bar{Q}/\underline{Q}, UB)$

3. Calculate the optimal $Q^* = (1-\omega)\underline{Q} + \omega\bar{Q}$ (if exist) and total transfer $\tilde{G}$

* $\tilde{g}^* \leq LB \rightarrow \tilde{G} = Q^*$
** $\tilde{g}^* \geq UB \rightarrow \tilde{G} = \tilde{g}^*$
*** $\tilde{g}^* \in (LB, UB) \rightarrow \tilde{G} = Q^*$

4. Evaluate both player's surplus and select the maximum total surplus